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Comments on "Strain-Displacement Relations in Large Displacement Theory of Shells"

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THE writer has recently reached similar conclusions to those of Tsao[1] concerning Novozhilov's[2] derivation of strain-displacement relations for the large displacement theory of thin shells under the assumptions that surface shears and elongations are negligible as compared to unity, and that the Kirchoff approximation on the behavior of the normal to the middle surface are valid. Equation (VI·26) of Ref. 2 contains an error and the approximate Eq. (VI·34) of Ref. 2 is open to question. However, it is not agreed that Eq. (VI·34) of Ref. 2 is erroneous in general, as stated by Tsao.[1] Tsao's conclusion is undoubtedly based on the fact that the strain-displacement relations of Ref. 2 do not reduce to those of the linear theory of shells developed by Novozhilov in Ref. 3. However, as pointed out in Chap. 1 of Ref. 2 and implied in the derivation in Chap. 1 of Ref. 3, the linear theory of elasticity is based on the assumptions that 1) the elongations, shears, and angles of rotation must be small as compared to unity, and 2) that the terms of the second degree in the angles of rotation appearing in the strain displacement relations as a result of the previous assumption must be small as compared to the corresponding strain components. The large displacement theory derived by Novozhilov[2] contains no explicit restrictions on the angles of rotation, although their relative magnitude in relation to the strain components is implied by Eq. (VI·34).[2] On the other hand, the method of derivation followed by Tsao[1] in proceeding from Eq. (7) to Eq. (12) of his paper also implicitly restricts the magnitude of the angles of rotation as compared to both the strain components and unity. Thus, neither Tsao[1] nor Novozhilov[2] provide general strain-displacement relations based on the Kirchoff approximations and the assumptions on shears and elongations alone, without restricting the angles of rotation otherwise, but both are correct within implied assumptions on the rotations.

Following the notation of Ref. 2 and introducing the function f, it is found that

$$f = 2(\bar{\epsilon}_{11} + \bar{\epsilon}_{22}) - \bar{\epsilon}_{12}^2 + 4\bar{\epsilon}_{11}\bar{\epsilon}_{22} \tag{1}$$

$$\alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1 + f \tag{2}$$

$$\chi = \alpha_{33}(1 + f)^{-1/2} - 1 \tag{3}$$

$$\theta = \alpha_{31}(1 + f)^{-1/2} \tag{4}$$

$$\psi = \alpha_{32}(1 + f)^{-1/2} \tag{5}$$

Since the middle surface strains $\bar{\epsilon}_{11}$, $\bar{\epsilon}_{22}$, and $\bar{\epsilon}_{12}$ are assumed small as compared to unity, it is seen that f is also small as compared to unity. If desired, the function $(1 + f)^{-1/2}$ can be expanded in ascending powers of these strains, such as

$$(1 + f)^{-1/2} = 1 - (\bar{\epsilon}_{11} + \bar{\epsilon}_{22}) + \tfrac{1}{2}(\bar{\epsilon}_{12}^2 + 2\bar{\epsilon}_{11}\bar{\epsilon}_{22} + 3\bar{\epsilon}_{11}^2 + 3\bar{\epsilon}_{22}^2) + \ldots \tag{6}$$

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When the terms shown in Eq. (6) are used in Eqs. (3–5), if desired, approximations to χ, θ, and ψ can be developed whose leading terms, in ascending powers of $\bar{u}$, $\bar{v}$, $\bar{w}$ and their derivatives, are those given by Eqs. (11) and (12) of Ref. 1. However, for a general theory within the confines of small strains and the Kirchoff approximations alone, the higher order terms in $\bar{u}$, $\bar{v}$, $\bar{w}$, and their derivatives so produced cannot be arbitrarily neglected as was done in Ref. 1. In fact, for a general nonlinear theory of thin shells there appears to be no need to reduce Eqs. (3–5) from their given form at this stage of the development of the theory.

It should be noted that the strain expansion terms given by Eq. (14) of Ref. 1, corrected with the unapproximated definitions of χ, θ, and ψ [Eqs. (3–5)] are analogous to the terms of Eqs. (4.24) of Ref. 3 for the linear theory, rather than a similarly corrected version otherwise following the development of Eqs. (VI·39) of Ref. 2 for the nonlinear theory. As stated also by both Refs. 2 and 3, there is no need to consider the terms v_{11}, v_{12}, and v_{22} multiplying z^2 in the strain expansions, Eqs. (13) of Ref. 1. In addition, in order that the thin-shell strain measures of the nonlinear theory reduce to those used in Ref. 1 in the linear case, ϵ_1, ϵ_2, ω, κ_1, κ_2 and τ, it is necessary to define the nonlinear strain measures as follows:

$$\left.\begin{aligned} \epsilon_1 &= \bar{\epsilon}_{11} & \kappa_1 &= -\chi_{11} + (\bar{\epsilon}_{11}/R_1) \\ \epsilon_2 &= \bar{\epsilon}_{22} & \kappa_2 &= -\chi_{22} + (\bar{\epsilon}_{22}/R_2) \\ \omega &= \bar{\epsilon}_{12} & 2\tau &= -\chi_{12} + (1/R_1 + 1/R_2)\bar{\epsilon}_{12} \end{aligned}\right\} \tag{7}$$

Equations (7) account for the differences of notation between Tsao[1] and Novozhilov.[2,3] The thin-shell strain measures given by Eqs. (7) are acceptable in the linear case as discussed, for example, by Budiansky and Sanders,[4] and in the nonlinear case should be acceptable in the same sense as those developed by Sanders[5] as an extension of the linear equations of Ref. 4.

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Erratum: "Expansion of a Finite Mass of Gas Into Vacuum"

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EQUATION (3) should read

$$C_1(\gamma) = 2^{(1-1/\lambda)}[\Gamma(2\nu - 1)/\Gamma(\nu)^2] \tag{3}$$

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